Calculating Value-at-Risk contributions in CreditRisk^{+*}

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Abstract

Credit Suisse First Boston (CSFB) launched in 1997 the model *CreditRisk*$^+$ which aims at calculating the loss distribution of a credit portfolio on the basis of a methodology from actuarial mathematics. Knowing the loss distribution, it is possible to determine quantile-based values-at-risk (VaRs) for the portfolio. An open question is how to attribute *fair* VaR contributions to the credits or loans forming the portfolio. One approach is to define the contributions as certain conditional expectations. We develop an algorithm for the calculations involved in this approach. This algorithm can be adapted for computing the contributions to the portfolio Expected Shortfall (ES).

1 Introduction

The CreditRisk$^+$ model (CSFB, 1997) has found wide-spread applications for the measurement of risk in credit portfolios. Reasons for the success of the model might be its free availability, the speed of calculations based on it due to fact that no Monte Carlo simulations need to be performed, and its detailed documentation. The basic idea in the model is to apply a methodology from actuarial mathematics.

As soon as the loss distribution has been computed, it is an easy task to calculate the portfolio value-at-risk (VaR). A further step might be to perform a risk diagnostics in the spirit of Litterman (1996). Such a diagnostics presupposes that the total portfolio risk can be attributed to the portfolio components in a way that detects the impact on the portfolio risk by the components.

A first suggestion for determining such VaR contributions was made in the CreditRisk$^+$ documentation (based on a decomposition of the portfolio variance (see CSFB, 1997); cf. also Bürgisser et al., 1999). Further proposals can be found in Lehrbass et al. (2001, sec. 5) or

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Martin et al. (2001). In both these latter proposals the VaR contributions are motivated as approximations to partial derivatives of VaR. Here we follow an alternative approach which relies on the fact that under certain continuity assumptions the partial derivatives of VaR with respect to the weights of the assets are just conditional expectations. This leads to an alternative definition of VaR contributions (see (2.7)) which might be useful for checking the results according to the approaches in Lehrbass et al. (2001) or Martin et al. (2001). Moreover, this definition can easily be adapted for the case of Expected Shortfall (ES) (see (4.2)) as risk measure.

This text is based on the representations of the CreditRisk$^+$ model as given by Lehrbass et al. (2001) and Gordy (2001). In section 2, we introduce some basic features of the CreditRisk$^+$ model and provide motivation for the kind of VaR contributions to be discussed below. Then, in section 3, we study the model in a more detailed manner in order to derive the main result of this paper, Corollary 3.4. It states that once it is possible to calculate the probability masses of the loss distributions, the computation of VaR contributions can be performed essentially the same way. This result is adapted to the case of ES in section 4. We conclude with a short summary of the results.

2 Abstract description of the problem

The key idea in CreditRisk$^+$. Consider a portfolio with n loans or credits. Default or non-default of loan i, $i = 1, \dots, n$, by a fixed time horizon is indicated by the random variable D_i which can only take the values 1 for default or 0 for non-default. Denote by ν_i the *exposure* of loan i, measured as a positive integer multiple of a certain currency unit. $p_i = \mathrm{P}[D_i = 1] \in (0, 1)$ is the *default probability* of loan i. The *realized* total loss L_0 of the portfolio is then given as

$$L_0 \;=\; \sum_{i=1}^{n} \nu_i\, D_i. \tag{2.1}$$

The following assumption is quite common for the purpose of dependency modeling in a portfolio of possible dependent assets (see Frey and McNeil, 2001). Suppose that all the random variables under consideration are defined on the underlying probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

Assumption 2.1 *There is a σ-algebra $\mathcal{A} \subset \mathcal{F}$ such that the default indicators $D_1, \dots, D_n$ are independent conditional on $\mathcal{A}$.*

We will see examples for the choice of $\mathcal{A}$ in section 3. $\mathcal{A}$ may be regarded as the collection of *systematic* factors influencing the portfolio.

If the default probabilities p_i are small, the total loss distribution might not change too much when the Bernoulli variables D_i in (2.1) are replaced by random variables $N_1, \dots, N_n$ with non-negative integer values that conditional on $\mathcal{A}$ are independent and Poisson-distributed with $\mathcal{A}$-measurable intensities $R_1, \dots, R_n > 0$ such that

$$\mathrm{E}[R_i] \;=\; p_i, \quad i = 1, \dots, n. \tag{2.2}$$

In the CreditRisk$^+$ model, one does not compute the distribution of the random variable L_0 from (2.1) but the distribution of the random variable

$$L \;=\; \sum_{i=1}^{n} \nu_i\, N_i \tag{2.3}$$

which, of course, is only an approximation to the true distribution.

Which risk contributions should be considered? For a fixed (close to 1) level $\delta \in (0,1)$,

$$\rho(L) \;=\; q_\delta(L) \;=\; \inf\{l \in \mathbb{R} : \mathrm{P}[L \le l] \ge \delta\} \tag{2.4}$$

is a widespread measure of portfolio loss risk. It is called *value-at-risk (VaR)* at level δ of L. In order to identify sources of particularly high risk in the portfolio, it is interesting to decompose $\rho(L)$ into a sum $\sum_{i=1}^n \rho_i(L)$ where the *risk contributions* $\rho_i(L)$ should correspond in some sense to the single losses $\nu_i\, N_i$.

Litterman (1996) suggested the decomposition

$$\rho(L) \;=\; \sum_{i=1}^{n} \frac{\partial \rho}{\partial h}\bigl(h\,\nu_i\, N_i + L\bigr)\Big|_{h=0}. \tag{2.5}$$

Decomposition (2.5) holds whenever the risk measure ρ is positively homogeneous (i.e. $\rho(h\,L) = h\,\rho(L)$ for $h > 0$) and differentiable. Unfortunately, the distribution of the portfolio loss L, specified by (2.3), is purely discontinuous. Therefore, the derivatives of $q_\delta(L)$ in the sense of (2.5) will either not exist or be 0.

Nonetheless, for a large portfolio even the positive probabilities $\mathrm{P}[L = l]$ will be rather small. On an appropriate scale, the distribution of L will therefore appear "almost" continuous. Tasche (1999) showed

- that Litterman's proposal (2.5) is the only decomposition which is compatible with RORAC (Return on risk-adjusted capital) based portfolio management, and
- that in case $\rho(L) = q_\delta(L)$ under certain continuity assumptions on the distribution of the asset losses one would have

$$\frac{\partial q_\delta}{\partial h}\bigl(h\,\nu_i\, N_i + L\bigr)\Big|_{h=0} \;=\; \nu_i\, \mathrm{E}[N_i \,|\, L = q_\delta(L)], \quad i = 1, \ldots, n. \tag{2.6}$$

Note that under the assumptions of Tasche (1999) the event $\{L = q_\delta(L)\}$ has probability 0. Hence, the conditional expectation in (2.6) has to be understood in the non-elementary sense (see Durrett, 1996, ch. 4).

Litterman's and Tasche's considerations suggest the definition of

$$\rho_i(L) \;=\; \nu_i\, \mathrm{E}[N_i \,|\, L = q_\delta(L)], \quad i = 1, \ldots, n, \tag{2.7}$$

as VaR contributions of the loans $i = 1, \ldots, n$ in the portfolio. Since $\mathrm{P}[L = q_\delta(L)]$ is positive by definition (2.4) of VaR, the conditional expectation in (2.7) is elementary in the sense that

$$\mathrm{E}[N_i \,|\, L = q_\delta(L)] \;=\; \frac{\mathrm{E}\big[N_i \,\mathbf{1}_{\{L = q_\delta(L)\}}\big]}{\mathrm{P}[L = q_\delta(L)]}, \tag{2.8}$$

where, as usual, the indicator function $\mathbf{1}_A$ denotes a random variable with value 1 on the event A and value 0 on the complement of A.

3 An algorithm for the VaR contributions

The aim with this section is to derive expressions for the VaR contributions defined by (2.7) which can be evaluated by the usual output of CreditRisk$^+$ or at least by minor modifications of this output.

In order to derive an expression for the expectation in (2.7) which can be numerically evaluated, we have to specify the σ-algebra from Assumption 2.1 and the stochastic intensities $R_1, \ldots, R_n$ from (2.2). This specification constitutes what is commonly called the CreditRisk$^+$ model. Recall that the Gamma-distribution with parameters $(\alpha, \beta) \in (0, \infty)^2$ is defined by the density $f_{\alpha,\beta}$ with

$$f_{\alpha,\beta}(x) \;=\; \frac{x^{\alpha-1}}{\beta^\alpha\,\Gamma(\alpha)}\, e^{-x/\beta}, \quad x > 0. \tag{3.1}$$

Assumption 3.1 *The stochastic intensities R_i of the N_i in (2.3), $i = 1, \ldots, n$, are given as*

$$R_i \;=\; \sum_{j=0}^{k} r_{j,i}\, S_j, \tag{3.2}$$

where $r_{j,i} \geq 0$, $\sum_{j=1}^{k} r_{j,i} > 0$, $\sum_{i=1}^{n} r_{j,i} = 1$, $S_0 \geq 0$ is a constant, and the $S_1, \ldots, S_k$ are independent and Gamma-distributed with parameters $(\alpha_j, \beta_j) \in (0, \infty)^2$, $j = 1, \ldots, k$. The σ-algebra $\mathcal{A}$ is the σ-algebra generated by $(S_1, \ldots, S_k)$, i.e. $\mathcal{A} = \sigma(S_1, \ldots, S_k)$. Eq. (2.2) holds.

Usually, the variables S_j are interpreted as systematic default intensities which are characteristic for sectors, countries, or branches. Equation (3.2) then mirrors the fact that a firm may be influenced by the evolutions in several sectors. Since $\mathrm{E}\big[S_j\big] = \alpha_j\,\beta_j$, $j = 1, \ldots, k$, under Assumption 3.1 eq. (2.2) is equivalent to

$$p_i \;=\; r_{0,i}\, S_0 + \sum_{j=1}^{k} r_{j,i}\, \alpha_j\, \beta_j, \quad i = 1, \ldots, n. \tag{3.3}$$

Remark 3.2

(i) *In practice, it is unlikely to have a credit portfolio parameterized like in Assumption 3.1. In CSFB (1997, sec. A12) and Lehrbass et al. (2001), for instance, the given data are the default probabilities p_i, $i = 1, \ldots, n$, and default volatilities σ_i, $i = 1, \ldots, n$, for each credit as well as factor weights $\theta_{j,i}$, $j = 0, \ldots, k$, $i = 1, \ldots, n$ which measure to which degree credit i is exposed to factor j and satisfy $\sum_{j=0}^{k} \theta_{j,i} = 1$, $i = 1, \ldots, n$. In this case, the $r_{j,i}$ are defined by*

$$r_{j,i} \;=\; \frac{p_i\, \theta_{j,i}}{\mu_j}, \tag{3.4a}$$

with $\mu_j = \sum_{i=1}^{n} p_i\, \theta_{j,i} = \mathrm{E}\big[S_j\big]$. Hence, S_0 is given by $S_0 = \mu_0$. It is quite common to specify also the variances τ_j^2 of the factors S_j, $j = 1, \ldots, k$ in order to determine the values of (α_j, β_j). Since $\mathrm{E}\big[S_j\big] = \alpha_j\, \beta_j$ and $\mathrm{var}\big[S_j\big] = \alpha_j\, \beta_j^2$, we have, once the expectations μ_j and the variances τ_j^2 are known,

$$\begin{aligned} \alpha_j &= \frac{\mu_j^2}{\tau_j^2}, \\ \beta_j &= \frac{\tau_j^2}{\mu_j}. \end{aligned} \tag{3.4b}$$

In CSFB (1997), the choice $\tau_j = \sum_{i=1}^{n} \theta_{j,i}\, \sigma_i$ is suggested in order to get a link between the default volatilities σ_i and the factor variances τ_j. However, this approach tends to underestimate the factor variances (see Kluge and Lehrbass, 2001). Therefore Kluge and Lehrbass propose to calculate the factor variances by

$$\tau_j \;=\; \sum_{i=1}^{n} \sqrt{\theta_{j,i}}\, \sigma_i, \quad j = 1, \ldots, k. \tag{3.4c}$$

(ii) *In contrast to Bürgisser et al. (1999) and Gordy (2001), we do* not *assume $\mathrm{E}[S_j] = 1$ in (3.2). Indeed, this would imply $\alpha_j \beta_j = 1$ and, as a consequence, would render the formulation of the main result in Corollary 3.4 quite difficult. In a self-explanatory mixture of our notions and the notions in Gordy (2001), Gordy's equation (1) and (3.2) above are related by*

$$\begin{aligned} r_{j,i} &= \frac{w_{ij}\, \bar{p}_i}{\sum_{l=1}^{n} w_{lj}\, \bar{p}_l}, \quad i = 1, \ldots, n, j = 0, \ldots, k, \\ \mathrm{E}[S_j] &= \sum_{l=1}^{n} w_{lj}\, \bar{p}_l, \quad j = 0, \ldots, k. \end{aligned} \tag{3.5}$$

From Assumption 3.1, we obtain the following representation of the generating function $g(z) = \mathrm{E}[z^L]$ of the distribution of L (cf. Gordy, 2001; Lehrbass et al., 2001)

$$g(z) \;=\; \exp\left(S_0\big(\sum_{i=1}^{n} r_{0,i}\, z^{\nu_i} - 1\big)\right) \prod_{j=1}^{k} \left(1 + \beta_j - \beta_j \sum_{i=1}^{n} r_{j,i}\, z^{\nu_i}\right)^{-\alpha_j}. \tag{3.6a}$$

Appealing to the binomial series it is clear that the radius of convergence ρ_g of the power series representation of g is given by

$$\rho_g \;=\; \sup\big\{z > 0 : \sum_{i=1}^{n} r_{j,i}\, z^{\nu_i} < 1 + \frac{1}{\beta_j},\, j = 1, \ldots, k\big\}. \tag{3.6b}$$

Obviously, we have $\rho_g > 1$, i.e. there are numbers $z > 1$ for which the power series of g converges. From (3.6a), the exact distribution of the portfolio loss L can be successively determined (see CSFB, 1997, A10) by means of a recurrence relation.

More important here, we can derive from (3.6a) the generating functions of the sequences $l \mapsto \mathrm{E}\big[N_i\, \mathbf{1}_{\{L=l\}}\big], i = 1, \ldots, n$.

Theorem 3.3 *Let $N_1, \ldots, N_n$ satisfy Assumption 3.1. Define L by (2.8), g by (3.6a), and ρ_g by (3.6b). Then the functions $f_i : (0, \rho_g) \to \mathbb{R}$, $i = 1, \ldots, n$ with*

$$f_i(z) \;=\; \sum_{l=0}^{\infty} \mathrm{E}\big[N_i\, \mathbf{1}_{\{L=l\}}\big]\, z^l \;=\; \mathrm{E}\big[N_i\, z^L\big] \tag{3.7a}$$

converge and can be represented as

$$f_i(z) \;=\; z^{\nu_i}\, g(z) \left(S_0\, r_{0,i} + \sum_{j=1}^{k} \frac{\alpha_j\, \beta_j\, r_{j,i}}{1 + \beta_j - \beta_j\, \sum_{h=1}^{n} r_{j,h} z^{\nu_h}} \right). \tag{3.7b}$$

Proof. Note that $\mathrm{E}\big[L\, z^L\big] = \sum_{l=0}^{\infty} l\, \mathrm{P}[L = l]\, z^l = z\, \frac{dg}{dz}(z)$ is finite for $z \in (0, \rho_g)$ as a consequence of general properties of power series. Since we assume $\nu_i > 0$, by

$$f_i(z) \;=\; \mathrm{E}\big[N_i\, z^L\big] \;\le\; \nu_i^{-1}\, \mathrm{E}\big[L\, z^L\big], \tag{3.8}$$

also the $f_i(z)$ are finite for $z \in (0, \rho_g)$. Fix $i \in \{1, \ldots, n\}$ and $z \in (0, \rho_g)$ and observe that

$$\frac{d}{d\,h}\big(z^{L+h\, N_i}\big)\big|_{h=0} \;=\; N_i\, z^L\, \log z. \tag{3.9}$$

Recall from section 2 that we assume the $\nu_1, \ldots, \nu_n$ to be positive integers. Nevertheless, (3.6a) holds not only for positive integers $\nu_1, \ldots, \nu_n$ but more generally for any $\nu_1, \ldots, \nu_n > 0$. Thus, for small h with $|h| < \nu_i$

$$\begin{aligned} \mathrm{E}\big[z^{L+h\, N_i}\big] \;=\; & \exp\left(S_0 \Big(\sum_{l=1, l\neq i}^{n} r_{0,l}\, z^{\nu_l} + r_{0,i}\, z^{\nu_i + h} - 1 \Big) \right) \\ & \times \prod_{j=1}^{k} \left(1 + \beta_j - \beta_j \left(\sum_{l=1, l\neq i}^{n} r_{j,l}\, z^{\nu_l} + r_{j,i}\, z^{\nu_i + h} \right) \right)^{-\alpha_j}. \end{aligned} \tag{3.10}$$

By (3.10), (3.9) will imply (3.7b) as soon as it is clear that the order of differentiation and expectation in $\frac{d}{d\,h}\mathrm{E}\big[z^{L+h\, N_i}\big]$ can be exchanged. But this follows from standard results on differentiation under the integral (e.g. Durrett, 1996, Theorem A.(9.1)). □

Note that Martin et al. (2001, translated to our notation) suggest the following approximation for the risk contributions to VaR in the sense of (2.7):

$$\rho_i(L) \;\approx\; \nu_i \, \frac{\mathrm{E}[N_i \, \exp(s_\delta \, L)]}{\mathrm{E}[\, \exp(s_\delta \, L)]}, \tag{3.11a}$$

where $s_\delta > 0$, the so-called *saddle point*, is given as the unique solution of the following equation

$$\frac{\mathrm{E}[L \, \exp(s_\delta \, L)]}{\mathrm{E}[\, \exp(s_\delta \, L)]} \;=\; \widehat{q}_\delta(L), \tag{3.11b}$$

with $\widehat{q}_\delta(L)$ standing either for $q_\delta(L)$ or any reasonable estimator of it. Note that the right-hand side of (3.11a) can be expressed as $\frac{1}{s_\delta} \frac{d}{dh} \log \mathrm{E}[\exp(z\,(L + h\,N_i))]\Big|_{(h=0,\, z=s_\delta)}$.

This approach is based on the so-called *saddle-point method* by means of which quantiles to tail probabilities of the distribution of L can be approximated (see Martin et al., 2001; Gordy, 2001).

Under Assumption 3.1, we obtain from Theorem 3.3 a rather simple formula for the right-hand side of (3.11a):

$$\rho_i(L) \;\approx\; \nu_i \, \exp(\nu_i \, s_\delta) \left(S_0 \, r_{0,i} + \sum_{j=1}^{k} \frac{\alpha_j \, \beta_j \, r_{j,i}}{1 + \beta_j - \beta_j \, \sum_{h=1}^{n} r_{j,h} \exp(\nu_h \, s_\delta)} \right). \tag{3.12}$$

In order to derive the announced algorithm for the calculation of the VaR contributions in (2.7), we need a further notation. Note that the probability measure P under consideration depends, in particular, on the parameters $\alpha_1, \ldots, \alpha_k$ from Assumption 3.1. We express this dependence by writing

$$\mathrm{P} \;=\; \mathrm{P}_\alpha \quad \text{with } \alpha = (\alpha_1, \ldots, \alpha_k),$$

and, analogously, E_α for the expectations.

Corollary 3.4 *Let $N_1, \ldots, N_n$ satisfy Assumption 3.1. Fix a confidence level $\delta \in (0, 1)$ and calculate the value-at-risk $q_\delta(L)$ of the portfolio loss L according to (2.4) for the probability P_α. Then, for $i = 1, \ldots, n$, the VaR contributions in the sense of (2.7) can be calculated by means of*

$$\mathrm{E}_\alpha[N_i \,|\, L = q_\delta(L)] \;=\; \frac{S_0 \, r_{0,i} \, \mathrm{P}_\alpha[L = q_\delta(L) - \nu_i] + \sum_{j=1}^{k} \alpha_j \, \beta_j \, r_{j,i} \, \mathrm{P}_{\alpha(j)}[L = q_\delta(L) - \nu_i]}{\mathrm{P}_\alpha[L = q_\delta(L)]}, \tag{3.13}$$

with $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\alpha(j) = (\alpha_1, \ldots, \alpha_j + 1, \ldots, \alpha_k)$.

Proof. Fix any $i \in \{1, \ldots, n\}$. By (2.8), it suffices to show that the numerator of the right-hand side of (3.13) equals $\mathrm{E}_\alpha[N_i \, \mathbf{1}_{\{L = q_\delta(L)\}}]$. We will prove the even stronger identity

$$\mathrm{E}_\alpha[N_i \, \mathbf{1}_{\{L=t\}}] \;=\; S_0 \, r_{0,i} \, \mathrm{P}_\alpha[L = t - \nu_i] + \sum_{j=1}^{k} \alpha_j \, \beta_j \, r_{j,i} \, \mathrm{P}_{\alpha(j)}[L = t - \nu_i]\,, \tag{3.14}$$

for any non-negative integer t. By Theorem 3.3 we know the generating function $f_i(z)$ of the sequence $t \mapsto \mathrm{E}_\alpha[N_i\, \mathbf{1}_{\{L=t\}}]$. Write g_α instead of just g in order to express the fact that g like P depends on the parameters $\alpha_1, \ldots, \alpha_k$ from Assumption 3.1. With this notation, (3.7b) can be written as

$$\begin{aligned} f_i(z) &= z^{\nu_i}\left(S_0\, r_{0,i}\, g_\alpha(z) + \sum_{j=1}^k \alpha_j\, \beta_j\, r_{j,i}\, g_{\alpha(j)}(z)\right) \\ &= \sum_{t=0}^\infty \left(S_0\, r_{0,i}\, \mathrm{P}_\alpha[L = t - \nu_i] + \sum_{j=1}^k \alpha_j\, \beta_j\, r_{j,i}\, \mathrm{P}_{\alpha(j)}[L = t-\nu_i]\right) z^t. \end{aligned} \tag{3.15}$$

Note that for the second identity in (3.15) we have used the fact that L is non-negative. By the uniqueness of power series, (3.7a) and (3.15) imply now (3.14). □

Remark 3.5

(i) *Corollary 3.4 states essentially that the VaR contributions according to (2.7) can be determined by calculating the loss distribution $(k + 1)$ times with different parameters. This is quite easy if the portfolio is parameterized as in Assumption 3.1. But also in the case described in Remark 3.2 (i) (with (3.4c) as definition for τ_j) it is easy to run the algorithm in the appropriate parameterizations. Just note that in order to calculate the $\mathrm{P}_{\alpha(j)}$ probabilities, it suffices to replace the weights $\theta_{j,i}$, $i = 1, \ldots, n$, with $\theta'_{j,i} = \frac{\alpha_j+1}{\alpha_j}\,\theta_{j,i}$.*

(ii) *By construction of N_i as a conditionally Poisson distributed random variable, we have $\mathrm{P}_\alpha[N_i > 1] > 0$. Hence it is possible that the VaR contributions according to (3.13) and (2.7) become greater than the exposures ν_i.*

(iii) *A further consequence of the model construction according to Assumption 3.1 is that*

$$\mathrm{E}_\alpha[N_i \,|\, L = q_\delta(L)] \;=\; 0 \tag{3.16}$$

may happen. In fact, for fixed $i \in \{1, \ldots, n\}$ the following statement holds:
$\mathrm{E}_\alpha[N_i \,|\, L = q_\delta(L)]$ equals 0 if and only if $\sum\limits_{j=1}^n \nu_j\, u_j = q_\delta(L)$ with non-negative integers $u_1, \ldots, u_n$ implies $u_i = 0$.
To see this, define $Y = L - \nu_i\, N_i$, and observe that

$$\begin{aligned} \mathrm{P}[L = q_\delta(L) - \nu_i] = 0 \quad &\iff \quad \mathrm{P}[Y = q_\delta(L) - k\,\nu_i] = 0,\ 1 \le k \le q_\delta(L)/\nu_i \\ &\iff \quad \mathrm{P}[L = q_\delta(L)] = \mathrm{P}[Y = q_\delta(L), N_i = 0] \\ &\iff \quad \{L = q_\delta(L)\} = \{Y = q_\delta(L), N_i = 0\}. \end{aligned} \tag{3.17}$$

Denote by M the set of all n-tuples $(u_1, \ldots, u_n)$ of non-negative integers such that $\sum_{j=1}^n \nu_j\, u_j = q_\delta(L)$. Then the assertion follows from (3.17) and the fact that

$$\{L = q_\delta(L)\} \;=\; \bigcup_{(u_1,\ldots,u_n)\in M} \{N_1 = u_1, \ldots, N_n = u_n\}. \tag{3.18}$$

In particular, (3.16) can occur in case $\nu_i > q_\delta(L)$. This is possible whenever $p_i < 1 - \delta$ and ν_i is large compared to the other exposures.

Remark 3.5 (ii) is being caused by the fact that CreditRisk$^+$ makes use of a Poisson approximation in order to compute the portfolio VaR. Hence one has to consider the Poisson variables N_i when computing risk contributions. As a consequence, there is little hope that switching to another risk measure will solve the problem of contributions which are larger than the corresponding exposures. The situation concerning Remark 3.5 (iii) is more favorable.

For instance, one can try to *smooth* the conditional expectations in (3.13) by choosing a fixed positive integer t and computing $\mathrm{E}_\alpha[N_i \,|\, q_\delta(L) - t \le L \le q_\delta(L) + t]$ instead of $\mathrm{E}_\alpha[N_i \,|\, L = q_\delta(L)]$. We will not go into the details of this approach since in the subsequent section we will see, that, in particular, the problem from Remark 3.5 (iii) can be avoided by switching from VaR to Expected Shortfall, a risk measure which should be preferred for some reasons (Acerbi and Tasche, 2001).

4 Contributions to Expected Shortfall

The *Expected Shortfall (ES)* at level $\delta \in (0,1)$ of the portfolio loss L can be defined as

$$\mathrm{ES}_\delta(L) \;=\; (1-\delta)^{-1} \int_\delta^1 q_u(L)\, du. \tag{4.1}$$

It may be characterized as the smallest *coherent* risk measure dominating VaR and only depending on L through its distribution (Delbaen, 1998, Th. 6.10). Of course, (4.1) appears not very handy for calculations or defining ES contributions. Nevertheless, if L had a continuous distribution, the representation

$$\mathrm{ES}_\delta(L) \;=\; \mathrm{E}[L \,|\, L \ge q_\delta(L)] \tag{4.2}$$

would be equivalent to (4.1) (Acerbi and Tasche, 2001, Cor. 5.3). Indeed, if we assume to deal with a large portfolio we can hope that the difference between (4.1) and (4.2) will not be too large. With (4.2) as definition, the decomposition

$$\mathrm{E}[L \,|\, L \ge q_\delta(L)] \;=\; \sum_{i=1}^n \nu_i\, \mathrm{E}[N_i \,|\, L \ge q_\delta(L)] \tag{4.3}$$

suggests the choice of $\nu_i\, \mathrm{E}[N_i \,|\, L \ge q_\delta(L)]$ as ES contribution of asset i. Further evidence for this choice stems from the fact under certain smoothness assumptions (4.3) is just the decomposition corresponding to (2.5) with $\rho = \mathrm{ES}$ (Tasche, 1999, Lemma 5.6).

From Corollary 3.4 we obtain the following result on the computation of the ES contributions.

Corollary 4.1 *Let $N_1, \ldots, N_n$ satisfy Assumption 3.1. Fix a confidence level $\delta \in (0,1)$ and calculate the value-at-risk $q_\delta(L)$ of the portfolio loss L according to (2.4) for the probability P_α. Then, for $i = 1, \ldots, n$, the ES contributions in the sense of (4.3) can be calculated by means of*

$$\mathrm{E}_\alpha[N_i \,|\, L \ge q_\delta(L)] \;=\; \frac{S_0\, r_{0,i}\, \mathrm{P}_\alpha[L \ge q_\delta(L) - \nu_i] + \sum_{j=1}^k \alpha_j\, \beta_j\, r_{j,i}\, \mathrm{P}_{\alpha(j)}[L \ge q_\delta(L) - \nu_i]}{\mathrm{P}_\alpha[L \ge q_\delta(L)]}, \tag{4.4}$$

with $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\alpha(j) = (\alpha_1, \ldots, \alpha_j + 1, \ldots, \alpha_k)$.

Proof. Observe that

$$\mathrm{E}_\alpha[N_i \,|\, L \geq q_\delta(L)] \;=\; \frac{\sum_{t=q_\delta(L)}^{\infty} \mathrm{E}_\alpha[N_i\, \mathbf{1}_{\{L=t\}}]}{\mathrm{P}_\alpha[L \geq q_\delta(L)]} \tag{4.5}$$

and note that in the proof of Corollary 3.4 we have in fact shown that

$$\mathrm{E}_\alpha[N_i\, \mathbf{1}_{\{L=t\}}] \;=\; S_0\, r_{0,i}\, \mathrm{P}_\alpha[L = t - \nu_i] + \sum_{j=1}^{k} \alpha_j\, \beta_j\, r_{j,i}\, \mathrm{P}_{\alpha(j)}[L = t - \nu_i] \tag{4.6}$$

for any non-negative integer t. □

Note that none of the probabilities in the numerator of the right-hand side of (4.4) can become 0, since by definition of $q_\delta(L)$ the probability $\mathrm{P}_\alpha[L \geq q_\delta(L) - \nu_i]$ satisfies

$$\mathrm{P}_\alpha[L \geq q_\delta(L) - \nu_i] \;\geq\; \mathrm{P}_\alpha[L \geq q_\delta(L)] \;\geq\; 1 - \delta \tag{4.7a}$$

and for any integer t we have

$$\mathrm{P}_\alpha[L = t] = 0 \quad \Longleftrightarrow \quad \mathrm{P}_{\alpha(j)}[L = t] = 0,\; j = 1, \ldots, k. \tag{4.7b}$$

Nonetheless, the observation in Remark 3.5 (ii) holds for the ES contributions as well.

5 Conclusion

Attributing the total risk of a credit portfolio to its components in an exhaustive and risk respecting way is an important task in portfolio management. For the case that risk is measured as value-at-risk (VaR) and determined with the CreditRisk$^+$ model (CSFB, 1997), we have shown that this attribution can be performed by calculating the loss distribution $(k+1)$ times (k denoting the number of sectors in the model) with slightly different parameters. The same statement holds for the attribution when risk is measured as Expected Shortfall (ES).

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